Integration of Price-Responsive Appliances in the Energy Market through Flexible Demand Saturation

Antonio De Paola, *Member, IEEE,* David Angeli, *Fellow, IEEE,* and Goran Strbac, *Member, IEEE*

***Abstract*—This paper proposes a novel decentralized technique for efficient integration of flexible demand in the electricity market. The analysis focuses on price-responsive appliances that schedule their power consumption on the basis of a demand/price signal received by a central entity. Previous work has shown that, when the devices population is sufficiently large to be described as a continuum, it is possible to provide necessary and sufficient conditions for the existence of a Nash equilibrium (no device has unilateral interest in changing its scheduling when considering the resulting profile of aggregate demand). These results are now extended in order to achieve an equilibrium also when the mentioned conditions are violated. To this purpose, a time-varying proportional constraint (equal for all devices) is introduced on the power rate of the price-responsive appliances so as to limit the variation of flexible demand that they can introduce at critical time instants. The proposed design technique not only guarantees existence of a Nash equilibrium but it also minimizes the global operation time of the appliances population. Simulation results are provided and it is shown that, under the considered assumptions, each individual appliance completes its task in minimum time.**

***Index Terms*—Electric power networks, flexible demand, game theory, distributed control.**

NOMENCLATURE

D	Broadcast demand signal (GW).
D_i	Inflexible demand profile (GW).
D_f	Flexible demand profile (GW).
D_a	Aggregate demand profile (GW).
D_r	Reference for flexible demand (GW).
Π	Energy price function ($£/KWh$).
Q_D	Cumulative distribution (h).
Λ_D	Negotiable valley capacity (GW/h).
Λ_f	Power density of task durations (GW/h).
E_{tot}	Energy required for task completion (KWh).
t_{min}	Minimum task time at rated power (h).
q_{max}	Max value of broadcast parameter t_{min} (h).
$\Gamma(t)$	Task time of appliances with $t_{min} = t$ (h).
$m(t,E)$	Unnormalized distribution of parameters t_{min} and E_{tot}.
$f'(t)$	Energy density of devices with $t_{min} = t$ (GW).
u^*	Scheduled profile of power consumption (KW).
α	Proportional constraint on max power consumption.
$\bar{F}$	Control input for demand shaping (GW/h).
$\bar{F}^*$	Task-time minimizing control (GW/h).
$\tilde{F}^*$	Optimal feedback control law (GW/h).
$(\bar{\alpha}_I, \bar{F}_I)$	States of forward dynamical system (h, GW).
$(\bar{\varphi}_\alpha, \bar{\varphi}_F)$	Solution of forward dynamical system (h, GW).
$(\tilde{\alpha}_I^\varepsilon, \tilde{F}_I^\varepsilon)$	States of backward dynamical system (h, GW).
$(\tilde{\varphi}_\alpha^\varepsilon, \tilde{\varphi}_F^\varepsilon)$	Solution of backward dynamical system (h, GW).
$(\tilde{\varphi}_\alpha, \tilde{\varphi}_F)$	Limiting solution of backward dynamical system (h, GW).
$\gamma^\varepsilon(\hat{T})$	Maximized final state $\tilde{\alpha}_I^\varepsilon(\hat{T})$ (h).
$\gamma(\hat{T})$	Limiting value of $\gamma^\varepsilon(\hat{T})$ (h).

A. De Paola, D. Angeli and G. Strbac are with the Department of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ UK (e-mail: ad5709@imperial.ac.uk; d.angeli@imperial.ac.uk; g.strbac@imperial.ac.uk).

I. INTRODUCTION

A clear trend in the evolution of power systems is the growing diffusion of new devices, such as "smart" appliances and electric vehicles, that have flexibility in the allocation of their power consumption. The potential benefits of this development are significant [1], [2]: electricity costs sustained by private customers could be reduced and the network could achieve an improved reliability, with a more efficient utilization of its assets. In order to fully realize this potential, it is crucial that the increasing amount of flexible demand is properly managed and coordinated. The main challenge is represented by the fact that, when the penetration of flexible demand in the system becomes significant, it is necessary to take into account the changes in demand and energy price introduced by the aggregate operation strategies of the appliances. For example, if all devices schedule their power consumption during the hours characterized by lower energy prices, the demand at those times will increase, altering the original price function and making their scheduling suboptimal. Different techniques have been proposed in the literature to tackle this problem, considering centralized mechanisms [3], distributed control strategies designed through stochastic optimization [4] and game theory [5]. Using the latter approach, we have proposed in [6] a decentralized scheduling for price-responsive devices. The single device, on the basis of a demand/price signal broadcast by some central entity, determines its power consumption in order to minimize its individual energy cost. By comparing two functions that abstract the properties of the power system and of the appliances population, we provide necessary and sufficient conditions for a Nash equilibrium. When these are satisfied, the power scheduling determined by each individual appliance on the basis of the broadcast signal (corresponding in general to the inflexible demand) is also optimal when the resulting aggregate demand is considered.

If the penetration of flexible demand in the system is above a certain threshold, the mentioned equilibrium conditions do not hold and it is not possible to determine a broadcast signal which ensures a stable system behaviour. In other words, the power scheduling of the devices will always be susceptible to renegotiation when the aggregate demand replaces the broadcast signal in the cost function of the individual agent. In practice, this would cause "rebound peaks" [7], [8]: the flexible appliances, trying to operate when energy is cheap, introduce a new peak demand which is potentially higher than the original one. This scenario corresponds to higher costs for the customers which now consume power at peak times. In addition, the power system is put under considerable stress as a result of high power transfers on transmission lines and use of costly fast generators.

A significant amount of research has focused on designing effective strategies to approach this issue. In order to limit the volatility and the instability phenomena in the system, [9] suggests the adoption of a pricing scheme that combines incremental updates of the electricity market price and stochastic elements. Pursuing similar objectives, [10] determines in an iterative fashion the price signal to be broadcast by the system operator in order to optimize the total utility of the agents. The introduction of a discontinuous price function is suggested in [11] to disincentivize the allocation of flexible demand after a certain threshold and limit the final energy price variation that the appliances can introduce. A stochastic technique is proposed in [12], broadcasting a randomized price to each appliance in order to avoid synchronicity of the power scheduling. With the same purpose, [13] suggests to introduce randomness on the controllers of the individual devices, considering at the same time an intermediate entity (aggregator) between the energy market and the individual customer. A stochastic distributed algorithm is also described in [14] for the specific case of electric vehicles, determining the charging profiles through an iterative procedure.

The approach proposed in the present work uses the theoretical analysis of [6] as a starting point and, through additional control actions that limit the flexible demand at critical time instants, achieves much more general results. In particular, the proposed technique always induces (under very general assumptions and for any level of flexible demand penetration) a Nash equilibrium in the electricity market. To do so, the flexible demand profile is reshaped through a proportional constraint on maximum power consumption, determined through the resolution of a Cauchy problem. By considering the differential equations in the opposite sense of integration and exploiting the monotonicity of the resulting dynamical system, it is possible to calculate the new constraint in order to induce an equilibrium and minimize the total time required by the appliances to complete their tasks. Similar results for large populations of flexible appliances (in particular, electric vehicles during charge) have been presented in [15] and [16]. These works show that, under certain assumptions, decentralized algorithms converge to a Nash equilibrium which is also socially optimal. We advocate that, in these cases, the cost function of the appliances includes artificial quadratic terms (deviation from previous iteration of the algorithm and from mean behaviour, respectively) that do not have real correspondence in the utility of the single devices. As a non-trivial consequence of their choice of utility function, the proposed methods guarantee convergence in a decentralized fashion to the social optimum, but this can be considered a Nash equilibrium only in a loose sense. In the present work we only consider the (linear) energy cost incurred by each device to complete its task. Moreover, convergence to a Nash equilibrium is guaranteed in one step and does not depend on the number of considered agents.

The rest of the paper is structured as follows: Section II summarizes the modeling choices and the main results presented in the previous work while Section III describes the optimization strategies and equilibrium conditions with saturated demand. The design technique for the proportional constraint is presented in Section IV and evaluated in simulations in Section V. The minimization of the individual task times with the proposed control scheme is proved in Section VI while Section VII contains some conclusive remarks.

II. Modelling and Unconstrained Nash Equilibria in the Energy Market

The main modelling elements and the equilibrium conditions provided in [6] are shortly summarized here for a clearer exposition of the additional results presented in this paper. It is assumed that each flexible appliance, at the beginning of the considered time interval $[0,T]$, broadcasts to a central entity the total energy E_{tot} required for its task and the minimum time t_{min} needed to complete it by operating at rated power $P_r = E_{tot}/t_{min}$. The sets of distinct values broadcast by the whole population for the required energy and minimum time are denoted by $\mathscr{E}$ and $\mathscr{T}$, respectively. If the appliances population is sufficiently large to be described by a continuum, it is possible to derive the unnormalized distribution $m(t_{min}, E_{tot})$ of the parameters, where $\int_{t_1}^{t_2}\int_{E_1}^{E_2} m(t,E)\,dE\,dt$ corresponds to the number of devices with $E_1 \le E_{tot} \le E_2$ and $t_1 \le t_{min} \le t_2$. The appliances population can then be characterized by the following functions:

$$f(t) = \int_0^t \int_{\mathscr{E}} E \cdot m(\tau, E)\,dE\,d\tau \qquad f'(t) = \int_{\mathscr{E}} E \cdot m(t,E)\,dE \tag{1}$$

where $f(t)$ represents the total amount of energy required by the appliances with $t_{min} \le t$.

Assumption 1: The function $f'(t) = \int_{\mathscr{E}} E \cdot m(t,E)\,dE$ is assumed of compact support:

$$\mathrm{supp}(f') = [q_{min}, q_{max}] \subset (0,T].$$

This assumption simplifies the notation and the subsequent analysis on Nash equilibria. Nevertheless, most of the presented results still hold if m is integrable and f' is a well-defined bounded function.

Remark 1: The derivative $f'(t)$ quantifies the total energy required by devices with $t_{min} = t$. Therefore, its support corresponds to the set $\mathscr{T}$ of distinct values for the parameter t_{min} taken by the whole population:

$$\mathscr{T} = \mathrm{supp}(f') = [q_{min}, q_{max}].$$

An interval for $\mathscr{T}$ may correspond to different typologies of appliances or, rather, represent the case of few homogeneous devices with different tasks, e.g. electric vehicles that require total charge and have different battery levels at $t = 0$.

The energy market is abstracted with the strictly monotone increasing function $\Pi : [0,+\infty) \to [0,+\infty)$ which returns, for a value of aggregate demand $D_a(t)$, the corresponding energy price $p(t) = \Pi(D_a(t))$. The aggregate profile D_a will be given by two distinct components: an inflexible profile D_i, known a priori, and the flexible component D_f which corresponds to the aggregate power profile of the appliances population. In the considered price-based program, the devices receive from the mentioned central entity a certain demand signal $D(t)$ (or equivalently the price $\Pi(D(t))$), and schedule their power consumption u in order to minimize their total energy cost:

$$\begin{aligned} \min_{u(\cdot)} \quad & \int_0^T \Pi(D(t)) \cdot u(t)\, dt \\ \text{s. t.} \quad & 0 \le u(t) \le \frac{E_{tot}}{t_{min}} \\ & \int_0^T u(t)\, dt = E_{tot} \end{aligned} \tag{2}$$

The analysis is restricted to a specific class of broadcast profiles D for which (2) has a unique optimal solution:

Assumption 2: The broadcast demand $D : [0,T] \to [0,+\infty)$ is a continuous function with no level sets of positive measure. For any $d \in \mathrm{Im}(D) = [d_{min}, d_{max}]$, it must hold:

$$\mu\left(\{\tau \in [0,T] : D(\tau) = d\}\right) = 0 \tag{3}$$

where μ represents the Lebesgue measure.

It has been shown in [6] that, under Assumption 2, the solution of (2) always exists and is unique up to sets of zero measure. It corresponds to operate at maximum feasible power during the t_{min} hours characterized by lowest demand D (and lowest price $\Pi(D)$). We now denote by $u^*_D(t,s,x)$ such solution as a function of time t, for devices with $t_{min} = s$ and $E_{tot} = x$. A Nash equilibrium is obtained if the power profile u^*_D, determined on the basis of the broadcast signal D, is still optimal when the resulting aggregate demand $D_{a,D}$ is considered. Equivalently, the following must hold for all $t_{min} = s \in \mathscr{T}$ and $E_{tot} = x \in \mathscr{E}$:

$$\begin{aligned} \int_0^T \Pi(D_{a,D}(t)) u^*_D(t,s,x) dt = \quad \min_{u(\cdot)} \quad & \int_0^T \Pi(D_{a,D}(t))) u(t)\, dt \\ \text{s. t.} \quad & 0 \le u(t) \le \frac{x}{s} \\ & \int_0^T u(t)\, dt = x \end{aligned} \tag{4}$$

The analysis is conducted by considering the sublevel sets of the broadcast demand profile. To this end, the following quantity is introduced:

Definition 1: Given a function D fulfilling Assumption 2, we define its *cumulative distribution* $Q_D : [d_{min}, d_{max}] \to [0,T]$ as:

$$Q_D(d) := \mu\left(\{\tau \in [0,T] : D(\tau) \le d\}\right). \tag{5}$$

Under Assumption 2, Q_D is continuous, strictly monotone increasing and takes the following values at the endpoints of its domain:

$$Q_D(d_{min}) = 0 \qquad Q_D(d_{max}) = T.$$

With the introduction of Q_D it is possible to replace the time t with the measure $q = Q_D(D(t))$ in our analysis, denoting the corresponding quantities in the new variable with a bar superscript. This allows the application of the theoretical tools and the extension of the equilibrium results presented in [6]. For a broadcast profile D fulfilling Assumption 2, the corresponding $\bar{D}$ is defined as:

$$\bar{D}(q) := Q_D^{-1}(q). \tag{6}$$

The quantity $\bar{D}(q)$ represents the demand value which induces a sublevel set of the broadcast signal D with measure q or, equivalently, such that $Q_D(\bar{D}(q)) = q$. Its relationship with the corresponding quantity in time is straightforward to obtain:

$$D(t) = \bar{D}(Q_D(D(t))) \qquad \forall t \in [0,T].$$

Remark 2: The representation in the variable q can be extended to any $\rho : [0,T] \to \mathbb{R}$ for which it holds:

$$\rho(t_1) = \rho(t_2) \quad \forall t_1, t_2 \in [0,T] : D(t_1) = D(t_2). \tag{7}$$

In this case, $\bar{\rho}(q)$ is equal to the function ρ evaluated at any time $t \in [0,T]$ such that $Q_D(D(t)) = q$. The following relationship holds between ρ and $\bar{\rho}$:

$$\rho(t) = \bar{\rho}(Q_D(D(t))) \qquad \forall t \in [0,T].$$

Two functions are now introduced: the *negotiable valley capacity* $\Lambda_D(q) = \frac{d}{dq} Q_D^{-1}(q)$, which describes the amount of flexible power that can be allocated in the valleys of the broadcast price signal while preserving an equilibrium, and the *power density of task durations* $\Lambda_f(q) = \frac{f'(q)}{q}$, which specifies how the appliances population will allocate their power consumption. The Nash equilibrium condition (4) is fulfilled (by choosing $D = D_i$) if and only if the following inequality is satisfied [6, Theorem 1-2]:

$$\Lambda_f(q) \le \Lambda_{D_i}(q) \qquad \forall q \in [q_{min}, q_{max}]. \tag{8}$$

III. Saturation of Flexible Demand

The possibility to extend the equilibrium conditions presented in [6] and summarized in the previous section is now investigated. When (8) does not hold, the sublevel sets of the broadcast profile and of the resulting aggregate demand do not correspond. This means that the power absorption of the flexible appliances introduces peaks in the aggregate profile at time instants when the broadcast demand is particularly low (and therefore energy is considered cheap). The resulting high energy prices at such peaks make the original scheduled profiles suboptimal for the aggregate demand and prevent the existence of a Nash equilibrium. To avoid this and limit the demand variation introduced by the flexible appliances at critical time instants, we consider an additional constraint on their maximum power absorption. The function $\alpha : [0,T] \to [0,1]$ is introduced for this purpose, defining a time-varying proportional constraint (equal for all the appliances) on the power consumption u. For devices with minimum task time t_{min}, total required energy E_{tot} and rated power $P_r = E_{tot}/t_{min}$, it must hold:

$$0 \le u(t) \le \alpha(t) \cdot \frac{E_{tot}}{t_{min}} \quad \forall t \in [0,T]. \tag{9}$$

It is supposed that (7) is satisfied for $\rho = \alpha$ and therefore it is possible to define the proportional constraint as a function $\bar{\alpha}(q)$ of the measure $q = Q_D(D(t))$, as specified in Remark 2.

The main idea is to design $\bar{\alpha}$ in order to modify the global behaviour of the appliances, shaping a new profile of flexible demand which always corresponds to a Nash equilibrium.

Remark 3: The proportional power constraint α can be directly enforced on the appliances. Alternatively, it can be implemented through discontinuous price functions, strongly penalizing those devices that exceed their prescribed fraction of power consumption. The fairness of this approach is guaranteed by Theorem 2, showing that the individual task times of all appliances can be minimized at once.

The power scheduling performed by each device on the basis of the broadcast D and the constraint α is discussed next. The design technique proposed for these quantities is presented in Section IV.

A. Optimal Scheduling of the Individual Device

The optimization problem solved by each appliance when a profile D is broadcast and the proportional constraint is introduced becomes:

$$\begin{aligned} \min_{u(\cdot)} \quad & \int_0^T \Pi(D(t)) \cdot u(t)\, dt \\ \text{s. t.} \quad & 0 \le u(t) \le \alpha(t) \frac{E_{tot}}{t_{min}} \\ & \int_0^T u(t)\, dt = E_{tot} \end{aligned} \tag{10}$$

In order to derive the optimal solution for (10), the minimization is analysed in the variable $q = Q_D(D(t))$. To this end, we introduce the following preliminary result:

Lemma 1: Under Assumption 2, given an integrable function $\rho : [0,T] \to \mathbb{R}$ that verifies (7) and therefore admits $\bar{\rho}$ as specified in Remark 2, we have:

$$\int_0^T \bar{\rho}(q)\, dq = \int_0^T \bar{\rho}\left(Q_D(D(t))\right) dt = \int_0^T \rho(t)\, dt. \tag{11}$$

The application of Lemma 1 for $\rho(t) = \Pi(D(t))u(t)$ allows to consider the equivalent problem in the q variable:

$$\begin{aligned} \min_{\bar{u}(\cdot)} \quad & \int_0^T \Pi(\bar{D}(q)) \cdot \bar{u}(q)\, dq \\ \text{s. t.} \quad & 0 \le \bar{u}(q) \le \bar{\alpha}(q) \frac{E_{tot}}{t_{min}} \\ & \int_0^T \bar{u}(q)\, dq = E_{tot} \end{aligned} \tag{12}$$

If one introduces the integral $\bar{\alpha}_I(q) = \int_0^q \bar{\alpha}(\tau)\, d\tau$ of the proportional constraint $\bar{\alpha}(q)$, it is possible to provide a closed-form expression for the solution of (12).

Proposition 1: Consider a profile D which fulfils Assumption 2 and $\bar{\alpha} : [0,T] \to [0,1]$ such that $\bar{\alpha}_I(T) = \int_0^T \bar{\alpha}(\tau)\, d\tau \ge q_{max}$. Problem (12) has a unique solution $\bar{u}_D^*(q,s,x)$ for appliances with $t_{min} = s \in \mathscr{T}$ and $E_{tot} = x \in \mathscr{E}$:

$$\bar{u}_D^*(q,s,x) = \begin{cases} \bar{\alpha}(q)\dfrac{x}{s} & \text{if} \quad \bar{\alpha}_I(q) \le s \\ \\ 0 & \quad\quad\;\; \bar{\alpha}_I(q) > s \end{cases} \tag{13}$$

Proof: For the feasibility of $\bar{u}_D^*$, consider that the inequalities in (12) hold by definition. Given that $\bar{\alpha}(q)$ takes values in $[0,1]$, the function $\bar{\alpha}_I(q)$ is continuous and nondecreasing. Since $\alpha_I(0) = 0$ and, from Remark 1, $t_{min} = s \le q_{max} \le \bar{\alpha}_I(T)$, there always exists a unique $\tilde{q}(s) \in [0,T]$ such that:

$$\bar{\alpha}_I(\tilde{q}(s)) = s$$
$$\bar{\alpha}_I(q) \le s \;\; \forall q \le \tilde{q}(s) \qquad \bar{\alpha}_I(q) > s \;\; \forall q > \tilde{q}(s).$$

Hence, the integral constraint in (12) is satisfied for $E_{tot} = x$:

$$\begin{aligned} \int_0^T \bar{u}_D^*(\tau,s,x)\, d\tau &= \int_0^{\tilde{q}(s)} \bar{u}_D^*(\tau,s,x)\, d\tau \\ &= \int_0^{\tilde{q}(s)} \bar{\alpha}(\tau)\frac{x}{s}\, d\tau = \bar{\alpha}_I(\tilde{q}(s))\frac{x}{s} = x. \end{aligned}$$

Consider that, as a result of the strict monotonicity of Π and (from Assumption 2) of Q_D and its inverse $\bar{D}(q) = Q_D^{-1}(q)$, the following inequalities hold:

$$\Pi(\bar{D}(q_1)) < \Pi(\bar{D}(q_2)) \qquad \forall q_1, q_2 \in [0,T] : q_1 < q_2.$$

To show that $\bar{u}_D^*$ is the unique minimizer for the objective function in (12) it is sufficient to notice that the total integral of any feasible control is fixed and $\bar{u}_D^*$ corresponds to the maximum feasible value of $\bar{u}$ in the interval $[0, \tilde{q}(s)]$. ■

We can conclude that, if the scheduling problem is considered in the variable $q = Q_D(D(t))$, each device will operate (at maximum feasible power $\bar{\alpha}(q) \cdot E_{tot}/t_{min}$) during an interval $[0, \tilde{q}(t_{min})]$ which corresponds to the lowest values of demand and ensures task completion since $\int_0^{\tilde{q}(t_{min})} \bar{u}_D^*(\tau, t_{min}, E_{tot})\, d\tau = E_{tot}$.

B. Characterization of Flexible and Aggregate Demand

Having calculated the power profile $\bar{u}_D^*$ scheduled by the single flexible device when D is broadcast, it is possible to derive the total demand variation $\bar{D}_f$ introduced by the appliances population as a function of the measure q. Taking the weighted integral of $\bar{u}_D^*$ over the set of broadcast parameters $t_{min} = s \in \mathscr{T}$ and $E_{tot} = x \in \mathscr{E}$ yields:

$$\begin{aligned} \bar{D}_f(q) &= \int_{\mathscr{E}} \int_{\mathscr{T}} \bar{u}_D^*(q,s,x) m(s,x)\, ds\, dx \\ &= \bar{\alpha}(q) \int_{\mathscr{E}} \int_{\bar{\alpha}_I(q)}^T \frac{x}{s} m(s,x)\, ds\, dx \\ &= \bar{\alpha}(q) \int_{\bar{\alpha}_I(q)}^T \frac{1}{s} \int_{\mathscr{E}} x m(s,x)\, dx\, ds = \bar{\alpha}(q) \int_{\bar{\alpha}_I(q)}^T \frac{f'(s)}{s}\, ds. \end{aligned} \tag{14}$$

Remark 4: The scheduled power profile u_D^* and flexible demand D_f can be obtained as functions of time by evaluating the corresponding expressions in the q variable at $q = Q_D(D(t))$:

$$u_D^*(t,s,x) = \bar{u}_D^*(Q_D(D(t)),s,x) \qquad D_f(t) = \bar{D}_f(Q_D(D(t))). \tag{15}$$

Optimality of u_D^* for problem (10) is straightforward to prove with the same arguments used in the proof of Proposition 1.

From Remark 4, it is possible to calculate the aggregate demand as a function of time:

$$D_{a,D_i}(t) = D_i(t) + D_f(t) = D_i(t) + \bar{D}_f\left(Q_D(D(t))\right). \tag{16}$$

Like the unconstrained case in [6], when the inflexible demand is broadcast ($D = D_i$) the aggregate demand can be expressed as a function of the current broadcast demand value $d = D_i(t)$:

$$\begin{aligned} D_{a,D_i}(t) = K(d) &= d + \bar{D}_f(Q_{D_i}(d)) \\ &= d + \bar{\alpha}(Q_{D_i}(d)) \int_{\bar{\alpha}_I(Q_{D_i}(d))}^T \frac{f'(s)}{s}\, ds. \end{aligned} \tag{17}$$

Remark 5: It is possible to provide an expression for the time $\Gamma(s)$ required by each appliance with parameter $t_{min}=s$ to complete its task when α and D are broadcast. From (13), considering (5) and Assumption 2, it holds:

$$\begin{aligned}\Gamma(s) &= \mu(\{t : \bar{\alpha}_I(Q_D(D(t))) < s\}) \\ &= \mu(\{q : \bar{\alpha}_I(q) < s\}) = \min_q \{q : \bar{\alpha}_I(q) \geq s\}\,.\end{aligned} \tag{18}$$

IV. Conditions for Nash Equilibria

In the following analysis, we assume that the demand signal D broadcast to the appliances is equal to the inflexible demand D_i. This assumption is motivated by the results provided in [6], where it is shown that $D=D_i$ is the only broadcast profile that can guarantee a Nash equilibrium in the unconstrained case. Moreover, this corresponds to a "rational" scheduling since the appliances population will allocate more power when the base profile D_i is lower and vice versa. The objective of this section is to design the constraint $\alpha(t)$ (or alternatively $\bar{\alpha}(q)$) in order to obtain a Nash equilibrium when condition (8) for the unconstrained case does not hold. This result is achieved if the power scheduling of each device, determined on the basis of the signal $D=D_i$, preserves its optimality when the resulting aggregate demand D_{a,D_i} is considered in the objective function of (10). Equivalently, the following must hold for all $t_{min}=s\in\mathscr{T}$ and $E_{tot}=x\in\mathscr{E}$:

$$\begin{aligned}\int_0^T \Pi(D_{a,D_i}(t))u^*_D(t,s,x)\,dt = \min_{u(\cdot)} \int_0^T \Pi(D_{a,D_i}(t))u(t)\,dt \\ \text{s. t. } 0\leq u(t)\leq \alpha(t)\frac{x}{s} \\ \int_0^T u(t)\,dt = x\end{aligned} \tag{19}$$

A preliminary result which considers the demand profiles as functions of the measure q is provided next. This is used in Section IV-A to approach the problem as a reshape of the flexible demand, calculating the corresponding constraint $\bar{\alpha}$ through the resolution of a Cauchy problem. By exploiting the monotonicity of the resulting dynamical system in the opposite sense of integration, as discussed in Section IV-B, we provide in Section IV-C a design method for $\bar{\alpha}$ which induces a Nash equilibrium and minimizes the total task time of the appliances. Such technique is summarized in its main phases by the scheme in Fig. 1.

Similarly to what has been presented in [6], it is possible to provide conditions in the variable q for which (19) is satisfied.

Proposition 2: For any integrable constraint function $\bar{\alpha}(\cdot)$ taking values in $[0,1]$ and broadcast demand $D=D_i$ fulfilling Assumption 2, the equilibrium condition (19) holds if:

$$\bar{D}_i'(q)+\bar{D}_f'(q)\geq 0 \qquad \forall q\in[0,T] \tag{20}$$

where $\bar{D}_i(q)$ denotes the inflexible demand as a function of the measure $q=Q_{D_i}(D_i(t))$, as specified in (6) with $D=D_i$, and $\bar{D}_f$ is the flexible demand profile defined in (14). Primes denote derivation with respect to the argument q.

Proof: From (13) and (15), devices with minimum time parameter t_{min} perform their task by operating at maximum feasible rate on the following interval:

$$\mathscr{S}_{D_i}(t_{min}) = \{t\in[0,T] : \bar{\alpha}_I(Q_{D_i}(D_i(t)))\leq t_{min}\}\,. \tag{21}$$

Given the monotonicity of Π, considering the objective function and the integral constraint in (19), the equilibrium condition (19) is verified if and only if the following holds for all $t_{min}\in\mathscr{T}$:

$$D_{a,D_i}(t_1)\leq D_{a,D_i}(t_2) \quad \forall t_1\in\mathscr{S}_{D_i}(t_{min})\ \forall t_2\in[0,T]\backslash\mathscr{S}_{D_i}(t_{min}). \tag{22}$$

Notice now that $\bar{\alpha}_I(q)$ and $Q_{D_i}(d)$ are monotone increasing functions. From expression (21) for $\mathscr{S}_{D_i}(t_{min})$, the values of inflexible demand at the time instants t_1 and t_2 considered above are such that $d_1=D_i(t_1)\leq D_i(t_2)=d_2$. Using the demand function K introduced in (17), the inequalities in (22) are verified if:

$$K(d_1)\leq K(d_2) \qquad \forall d_1,d_2\in[d_{min},d_{max}] : d_1\leq d_2. \tag{23}$$

This is equivalent to impose nonnegativity of $K'(d)$:

$$K'(d) = 1+\bar{D}_f'(Q_{D_i}(d))Q_{D_i}'(d)\geq 0. \tag{24}$$

Dividing both terms of the inequality by $Q_{D_i}'(d)$ and letting q denote $Q_{D_i}(d)$ yields:

$$\left(\frac{d}{dq}Q_{D_i}^{-1}(q)\right)+\bar{D}_f'(q) = \bar{D}_i'(q)+\bar{D}_f'(q)\geq 0. \tag{25}$$

■

A. Shaping of Flexible Demand

The design of a constraint function $\bar{\alpha}$ for which (20) holds is nontrivial since its relationship with $\bar{D}_f'$ is not instantaneous:

$$\bar{D}_f'(q) = \bar{\alpha}'(q)\int_{\bar{\alpha}_I(q)}^T \frac{f'(s)}{s}\,ds - \bar{\alpha}^2(q)\frac{f'(\bar{\alpha}_I(q))}{\bar{\alpha}_I(q)}. \tag{26}$$

For this reason, a desired profile of flexible demand that fulfils the equilibrium condition of Proposition 2 will be initially calculated, deriving only as a second step the function $\bar{\alpha}$ needed to generate such profile. In particular, an additional function $\bar{F}:[0,T]\to\mathbb{R}^+$ and a reference $\bar{D}_r$ for the flexible demand are introduced with:

$$\bar{D}_r(q) = \int_q^T \bar{F}(\tau)\,d\tau. \tag{27}$$

For a given function $\bar{F}(\cdot)$, it is possible to define the following dynamical system with states $\bar{\alpha}_I$ and $\bar{F}_I$:

$$\begin{aligned}&\dot{\bar{\alpha}}_I(q) = \bar{\alpha}(q) = \frac{\int_q^T \bar{F}(\tau)\,d\tau}{\int_{\bar{\alpha}_I(q)}^T \frac{f'(\tau)}{\tau}\,d\tau} = \frac{F_{tot}-\bar{F}_I(q)}{\int_{\bar{\alpha}_I(q)}^T \frac{f'(\tau)}{\tau}\,d\tau} \quad && \dot{\bar{F}}_I(q) = \bar{F}(q) \\ &\bar{\alpha}_I(0)=0 && \bar{F}_I(0)=0\end{aligned} \tag{28}$$

where $F_{tot}=\bar{F}_I(T)=\int_0^T \bar{F}(\tau)\,d\tau$ denotes the integral of $\bar{F}$ over the interval $[0,T]$. Given a feasible control $\bar{F}(\cdot)$, the unique solution of (28) will be denoted by $(\bar{\varphi}_\alpha(\cdot),\bar{\varphi}_F(\cdot))$. The definition of the derivative $\dot{\bar{\alpha}}_I(q)=\bar{\alpha}(q)$ guarantees that the resulting flexible demand $\bar{D}_f$, defined in (14), is equal to $\bar{D}_r$. Furthermore, for $\bar{D}_f=\bar{D}_r$, the equilibrium condition (20) becomes:

$$\bar{F}(q)\leq \bar{D}_i'(q) \qquad \forall q\in[0,T]. \tag{29}$$

Rather than directly calculating $\bar{\alpha}$, we determine $\bar{F}$ which satisfies (29) (and therefore guarantees an equilibrium), obtaining the corresponding $\bar{\alpha}$ through (28). In this respect, an additional constraint must be taken into account. Since $\bar{\alpha}(q)$ represents a proportional reduction in the maximum power of the devices, a state $(\bar{\alpha}_I(q),\bar{F}_I(q))$ will be feasible only if:

$$0 \le \bar{\alpha}(q) = \frac{F_{tot} - \bar{F}_I(q)}{\int_{\bar{\alpha}_I(q)}^{T} \frac{f'(\tau)}{\tau} d\tau} \le 1. \tag{30}$$

When determining $\bar{F}(\cdot)$ we do not only seek to satisfy the Nash equilibrium condition (29) but we also aim at optimizing some global properties of the system. In particular, we are interested in minimizing the total time required by the appliances to perform their tasks. It is shown in Section VI that this optimization guarantees minimum task duration also for the single devices. From (13) and Remark 4, the interval of power consumption for a device with $t_{min} = s$ is equal to $\mathscr{S}_D(s) = \{t \in [0,T] : \bar{\alpha}_I(Q_D(D(t)) < s)\}$. Given the monotonicity of $\bar{\alpha}_I$ and Q_D, we have that $\mathscr{S}_D(s_1) \subset \mathscr{S}_D(s_2)$ when $s_1 < s_2$. Therefore, the total task time of the population corresponds to $\Gamma(q_{max}) = \mu(\mathscr{S}_D(q_{max}))$. Considering expression (18) for Γ, the optimization problem can be defined as:

$$\begin{aligned} \min_{\bar{F}(\cdot),F_{tot},T_{END}} \quad & T_{END} \\ \text{s.t.} \quad & \bar{\alpha}_I(0) = 0 \qquad\qquad \bar{F}_I(0) = 0 \\ & \bar{\alpha}_I(T_{END}^-) = q_{max} \qquad \bar{F}_I(T_{END}^-) = F_{tot} \\ & \dot{\bar{\alpha}}_I(q) = \frac{F_{tot} - \bar{F}_I(q)}{\int_{\bar{\alpha}_I(q)}^{T} \frac{f'(\tau)}{\tau} d\tau} \qquad \dot{\bar{F}}_I(q) = \bar{F}(q) \\ & 0 \le \dot{\bar{\alpha}}_I(q) \le 1 \qquad\qquad \bar{F}(q) \le \bar{D}_i'(q) \\ & \forall q \in [0, T_{END}) \end{aligned} \tag{31}$$

The derivative constraints in (31) are considered on the interval $[0,T_{END})$ as $\dot{\bar{\alpha}}_I(q)$ is not well defined at $q = T_{END}$ if $\bar{\alpha}_I(T_{END}) = q_{max}$. In this case, numerator and denominator in its expression are equal to zero (for the latter term, consider that $\int_{\bar{\alpha}_I(q)}^{T} \frac{f'(\tau)}{\tau} d\tau = \int_{\bar{\alpha}_I(q)}^{q_{max}} \frac{f'(\tau)}{\tau} d\tau$ from Assumption 1). It is worth noticing that (31) corresponds to a minimum-time optimal control problem of a nonlinear system with input and state constraints. Therefore, it is not possible to determine a priori existence of its solution. The analysis in the next subsections will allow to prove in Theorem 1 that an optimal control $\bar{F} = \bar{F}^*$ exists, providing also its closed-form expression.

Remark 6: Once the minimization problem has been solved, the corresponding proportional constraint is equal to $\bar{\alpha}(q) = \dot{\bar{\alpha}}_I(q)$ of the optimal solution. It is straightforward to calculate the values of α in the time variable t with $\alpha(t) = \bar{\alpha}(Q_{D_i}(D_i(t)))$. Since all devices complete their task for $q \le T_{END}$, the values of $\bar{\alpha}(q)$ can be arbitrarily defined (for example equal to 1) when $q > T_{END}$.

B. Backward-integrated Dynamical System

One of the main challenges in the resolution of (31) is that the final value F_{tot} of the state $\bar{F}_I$, corresponding to the total integral of the control $\bar{F}(\cdot)$, appears in the dynamics of $\bar{\alpha}_I(q)$. For this reason, a different system is introduced in order to model the same dynamics of (28) in the opposite direction of integration. It will be shown that, for certain conditions on controls and final states, the solutions of the two systems are related and therefore it is possible to solve the optimization problem (31) without directly operating on (28). The new system is described by the following equations:

$$\begin{aligned} \dot{\tilde{\alpha}}_I^{\varepsilon}(q) &= \frac{\tilde{F}_I^{\varepsilon}(q)}{\int_{q_{max}-\tilde{\alpha}_I^{\varepsilon}(q)}^{q_{max}} \frac{f'(\tau)}{\tau} d\tau} = \frac{\tilde{F}_I^{\varepsilon}(q)}{h(\tilde{\alpha}_I^{\varepsilon}(q))} \qquad & \dot{\tilde{F}}_I^{\varepsilon}(q) &= \tilde{F}(q) \\ \tilde{\alpha}_I^{\varepsilon}(0) &= \varepsilon & \tilde{F}_I^{\varepsilon}(0) &= 0 \end{aligned} \tag{32}$$

where the function $h(x) = \int_{q_{max}-x}^{q_{max}} \frac{f'(\tau)}{\tau} d\tau$ is used for a more compact expression of $\dot{\tilde{\alpha}}_I^{\varepsilon}$. Notice that (32) defines a family of Cauchy problems parametrized by the initial condition ε of one of the state variables. Fixed a control profile $\tilde{F}(\cdot)$, the unique solution of (32) will be denoted by $(\tilde{\varphi}_{\alpha}^{\varepsilon}(\cdot), \tilde{\varphi}_F^{\varepsilon}(\cdot))$. Taking into account that $\dot{\tilde{\alpha}}_I^{\varepsilon}$ is not well defined when $\left(\tilde{\alpha}_I^0(0), \tilde{F}_I^0(0)\right) = (0,0)$, we consider decreasing values of ε, denoting by $(\tilde{\varphi}_{\alpha}(\cdot), \tilde{\varphi}_F(\cdot))$ the limit of solutions of (32) for ε that tends to zero. Such a limit exists and is unique as equation (32) defines a cooperative system (monotonicity with respect to initial conditions). Notice also that the $\tilde{\varphi}_F^{\varepsilon}$ component is actually independent of ε. It is now possible to provide a first result on the relationship between the state trajectories of the two discussed dynamical systems (28) and (32):

Proposition 3: Consider any $\bar{F}(\cdot)$ defined on $[0,\hat{T}]$, $\hat{T} > 0$, which is feasible for (28) and such that, for the corresponding solution $(\bar{\varphi}_{\alpha}(\cdot), \bar{\varphi}_F(\cdot))$, we have:

$$\bar{\varphi}_{\alpha}(\hat{T}^-) = q_{max} \qquad\qquad \bar{\varphi}_F(\hat{T}^-) = F_{tot} \tag{33}$$

Denote now by $\tilde{F}(\cdot)$ the control input of system (32) defined by $\tilde{F}(q) = \bar{F}(\hat{T} - q)$ for all $q \in [0,\hat{T}]$. Given the corresponding limiting solution $(\tilde{\varphi}_{\alpha}(\cdot), \tilde{\varphi}_F(\cdot))$, the following holds for all $q \in (0,\hat{T}]$:

$$\bar{\varphi}_{\alpha}(\hat{T} - q) = q_{max} - \tilde{\varphi}_{\alpha}(q) \qquad\qquad \bar{\varphi}_F(\hat{T} - q) = F_{tot} - \tilde{\varphi}_F(q) \tag{34}$$

Proof: See Appendix A. ■

A similar property holds in the opposite direction:

Proposition 4: Consider any $\tilde{F}(\cdot)$ defined on $[0,\hat{T}]$ which is feasible for (32) and such that the corresponding solution $(\tilde{\varphi}_{\alpha}(\cdot), \tilde{\varphi}_F(\cdot))$ when ε tends to zero satisfies the following conditions:

$$\tilde{\varphi}_{\alpha}(\hat{T}) = q_{max} \qquad\qquad \tilde{\varphi}_F(\hat{T}) = F_{tot} \tag{35}$$

If one denotes by $\bar{F}(\cdot)$ the control input of system (28) such that $\bar{F}(q) = \tilde{F}(\hat{T} - q)$, the resulting solution $(\bar{\varphi}_{\alpha}(\cdot), \bar{\varphi}_F(\cdot))$ fulfills the following equations for all $q \in [0,\hat{T})$:

$$\tilde{\varphi}_{\alpha}(\hat{T} - q) = q_{max} - \bar{\varphi}_{\alpha}(q) \qquad\qquad \tilde{\varphi}_F(\hat{T} - q) = F_{tot} - \bar{\varphi}_F(q) \tag{36}$$

Proof: See Appendix B. ■

Proposition 3 and 4 show the correspondence between the state trajectories of the dynamical systems (28) and (32) if certain conditions are verified for the control inputs and final states. In the next subsection an optimization will be performed on the states of (32), using the equivalent for system (28) of the resulting optimal control to solve the time minimization problem (31) and induce a Nash equilibrium. The choice to operate on system (32) is motivated not only by the dependency of $\dot{\bar{\alpha}}_I$ in the original system (28) from the final state F_{tot} but also by its monotonicity properties:

Proposition 5: The dynamical system described by (32) is cooperative [17].

Proof: It is sufficient to consider the sign of the following partial derivatives:

$$\begin{array}{ll}\dfrac{\partial\dot{\tilde{\alpha}}_I^\varepsilon}{\partial\tilde{F}_I^\varepsilon}=\dfrac{1}{h\left(\tilde{\alpha}_I^\varepsilon\right)}\geq 0 & \dfrac{\partial\dot{\tilde{F}}_I^\varepsilon}{\partial\tilde{\alpha}_I^\varepsilon}=0\\ \dfrac{\partial\dot{\tilde{\alpha}}_I^\varepsilon}{\partial\tilde{F}}=0 & \dfrac{\partial\dot{\tilde{F}}_I^\varepsilon}{\partial\tilde{F}}=1>0\end{array} \tag{37}$$

■

C. Task-time Minimizing Solution

The sets of admissible states $\left(\tilde{\alpha}_I^\varepsilon,\tilde{F}_I^\varepsilon\right)$ and controls $\tilde{F}(\cdot)$ for system (32), respectively $\mathscr{X}$ and $\mathscr{U}_{\hat{T}}$, are defined as follows:

$$\begin{aligned}\mathscr{X} &= \left\{\left(\tilde{\alpha}_I^\varepsilon,\tilde{F}_I^\varepsilon\right):\tilde{F}_I^\varepsilon\leq h\left(\tilde{\alpha}_I^\varepsilon\right)\right\}\\ \mathscr{U}_{\hat{T}} &= \left\{\tilde{F}(\cdot):\tilde{F}(q)\in\left[0,\bar{D}_i'(\hat{T}-q)\right]\ \forall q\in[0,\hat{T}]\right\}.\end{aligned} \tag{38}$$

From Proposition 5, system (32) is monotone for the orders induced from the positive orthants in the state and control space. Therefore, it is possible to maximize its state components by applying, at each q, the maximum feasible control. Each value of $\hat{T}$ induces a corresponding maximizing solution. With the proper choice of the parameter $\hat{T}$, it is then possible to satisfy (35) and apply Proposition 4, extending the same result to the forward system (28) and allowing to solve the original optimization problem (31). In order to do so, reminding that $h(x)=\int_{q_{max}-x}^{q_{max}}\frac{f'(\tau)}{\tau}\,d\tau$, the following feedback law as a function of q and current states $\tilde{\alpha}_I$ and $\tilde{F}_I$ is introduced:

$$\tilde{F}^*(q,\tilde{\alpha}_I,\tilde{F}_I)=\begin{cases}\bar{D}_i'(\hat{T}-q) & \text{if}\quad \tilde{F}_I<h\left(\tilde{\alpha}_I\right)\\ \min\left(\bar{D}_i'(\hat{T}-q),h'\left(\alpha_I\right)\right) & \text{if}\quad \tilde{F}_I=h\left(\tilde{\alpha}_I\right)\end{cases} \tag{39}$$

We denote by $\Phi^{\hat{T}}(x_0,q)$ the solution of the ODEs in (32) at 'time' q and with initial conditions x_0 ($q=0$) when the (time-varying and discontinuous) feedback law $\tilde{F}^*$ is applied. The value function γ^ε of the following optimization problem and its limit when ε tends to zero are defined as:

$$\gamma^\varepsilon(\hat{T}):=\max_{\tilde{F}(\cdot)\in\mathscr{U}_{\hat{T}}}\tilde{\alpha}_I^\varepsilon(\hat{T})\qquad\qquad \gamma(\hat{T}):=\lim_{\varepsilon\to 0}\gamma^\varepsilon(\hat{T}) \tag{40}$$

Considering the monotonicity of system (32) from Proposition 5 and the fact that $\tilde{F}^*$ represents the maximum feasible control at any time instant and current state, for any $\hat{T}\geq 0$ it holds:

$$\gamma^\varepsilon(\hat{T})=\Phi_\alpha^{\hat{T}}([\varepsilon,0],\hat{T}) \tag{41}$$

where $\Phi_\alpha^{\hat{T}}$ denotes the component of $\Phi^{\hat{T}}$ corresponding to $\tilde{\alpha}_I$. Moreover, it is possible to show that the maximized final state $\gamma^\varepsilon(\hat{T})$ increases if we choose higher values of $\hat{T}$:

Proposition 6: The function $\gamma^\varepsilon(\hat{T})$ is Lipschitz continuous and monotone increasing.

Proof: See Appendix C. ■

It is now possible to provide the main result of this section, describing the solution of the optimization problem (31):

Theorem 1: If problem (31) is feasible, there exists $T^*\in[0,T]$ defined as the minimum t such that $\gamma(t)=q_{max}$. Denote now by $\tilde{\psi}^*(q)$ the following signal:

$$\tilde{\psi}^*(q)=\lim_{\varepsilon\to 0}\tilde{F}^*\left(q,\Phi^{T^*}([\varepsilon,0],q)\right). \tag{42}$$

The control $\bar{F}^*$ defined below is feasible and optimal for (31):

$$\bar{F}^*(q)=\tilde{\psi}^*(T^*-q)\quad\forall q\in[0,T^*]. \tag{43}$$

Proof: See Appendix D. ■

From the results of Theorem 1, we can conclude that the task-time minimizing profile of flexible demand in the q variable $\bar{D}_f^*(q)$ (and the function $\bar{\alpha}^*(q)$ that induces it when broadcast to the devices) can be calculated operating "backward". Fixed a certain final instant $\hat{T}$, we evaluate the feedback control $\tilde{F}^*$ of the backward system (32), maximizing its final state $\tilde{\alpha}_I(\hat{T})$. There exists $\hat{T}$ (equal to T^* in the theorem statement) such that $\tilde{\alpha}_I(\hat{T})=q_{max}$. This guarantees a correspondence with the forward system, as specified by Proposition 4. By considering the equivalent control $\bar{F}^*(q)=\tilde{F}^*(\hat{T}-q)$ forward in time, we solve the optimization problem (31). This means that the resulting value $\bar{\alpha}^*(q)=\dot{\bar{\alpha}}_I^*(q)$ in system (28) is the proportional constraint in the variable q that, if broadcast to the devices, achieves a Nash equilibrium and minimizes the total task time of the population. Since $\bar{\alpha}$ has been set in order to guarantee that $\bar{D}_f$ in (14) is equal to $\bar{D}_r$ in (27), the expression for the flexible demand induced by the optimal solution of (31) is:

$$\bar{D}_f^*(q)=\int_q^{T^*}\bar{F}^*(s)\,ds. \tag{44}$$

Remark 7: In practical implementations of the proposed control strategy, the final time T^* (whose existence is guaranteed by Theorem 1) can be easily calculated numerically. It is sufficient to iteratively integrate (32) with $\tilde{F}=\tilde{F}^*$, updating at each step the chosen value of $\hat{T}$. Exploiting the positivity of the state derivatives in (32) and the monotonicity of γ, the value of $\hat{T}$ at each step can be chosen with a bisection technique, obtaining a quick convergence to the sought value T^*.

A summary of the main steps in the implementation of the proposed control strategy is provided by the scheme in Fig. 1.

V. Simulation Results

The design method of the proportional constraint α presented in the previous section and summarized in Fig. 1 is now tested in simulations. A typical 24h UK demand profile [18] is chosen for D_i, adopting a time discretization step of $\Delta t=0.01h$. We consider a case study for which the equilibrium conditions in the unconstrained case, introduced in [6], are not satisfied. In particular, the function f' is defined as the sum of two truncated gaussians (with mean equal to $4h$ and $8h$) and the total energy required by the appliances amounts to $55GWh$. This choice could correspond, for example, to a population of 2 million devices (approximated as infinite) with equal power rating $P_r=5KW$ and an average value of E_{tot} corresponding to $\bar{E}=55GWh/2\cdot 10^6=27.5KWh$. The values of E_{tot} for the single agents would follow a distribution similar to the one considered for f', with peaks at $E_1=4h\cdot P_r=20KWh$

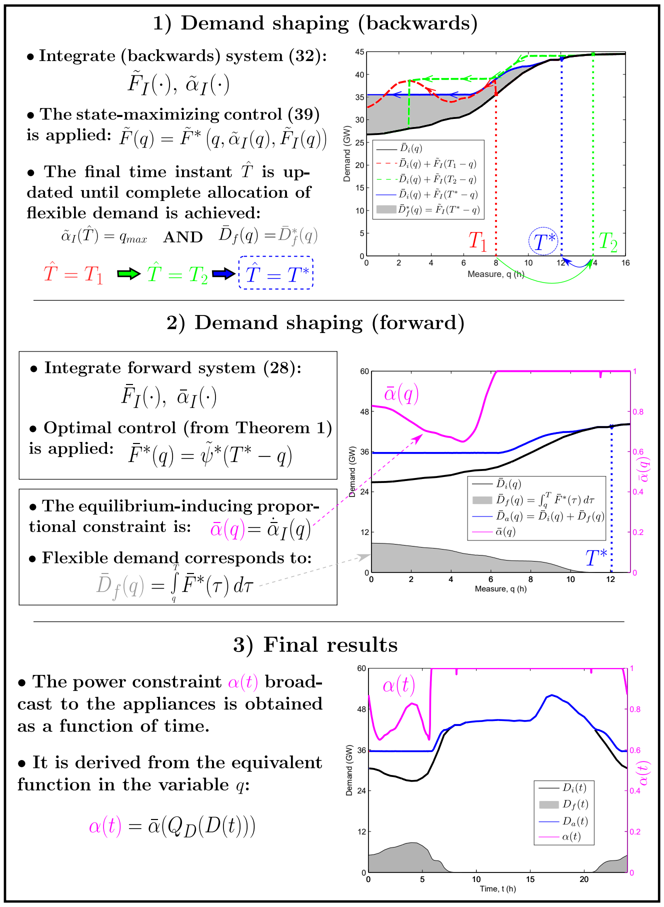

Fig. 1. Procedure to determine the optimal power constraint $\alpha(t)$.

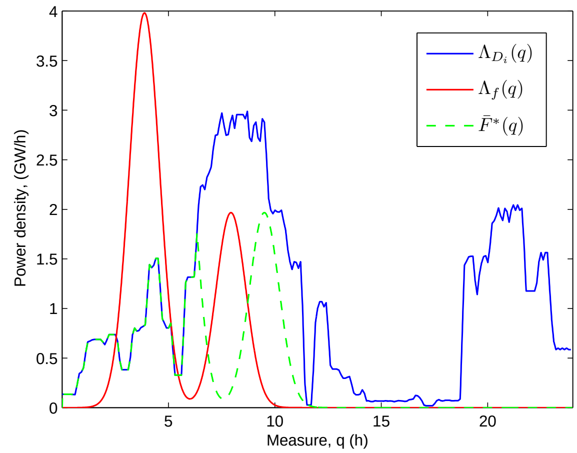

Fig. 2. Graphical representation of condition (8) for unconstrained equilibrium, showing negotiable valley capacity Λ_{D_i} (blue trace) and power density of task durations Λ_f (red trace) for the chosen f' and inflexible demand D_i. The green line is the optimal control $\bar{F}^*$ for the minimization problem (31).

and $E_2 = 8h \cdot P_r = 40KWh$. The corresponding time parameter could then be derived as $t_{min} = E_{tot}/P_r$. The negotiable valley capacity $\Lambda_{D_i}(q) = \frac{d}{dq} Q_{D_i}^{-1}(q)$ and the power density of task durations $\Lambda_f(q) = \frac{f'(q)}{q}$ for this scenario (defined at the end of Section II) are compared in Fig. 2. It is straightforward to verify that a Nash equilibrium cannot be achieved in the unconstrained case since (8) does not hold and $\Lambda_f(q) > \Lambda_{D_i}(q)$ in the interval which goes approximatively from $q = 2h$ to $q = 5h$. Therefore, a proportional constraint α is introduced on the power rate of the devices. This is calculated applying the design technique presented in the previous section, as summarized in Fig. 1. In order to do so, it is necessary to evaluate γ and $\tilde{\psi}^*$, defined in (40) and (42) as limiting functions for the initial state value ε which tends to zero. These quantities are approximated by evaluating the mentioned functions at $\varepsilon = 10^{-6}$. It has been verified that the choice of smaller values does not introduce any noticeable change in the results. For the considered scenario, setting an error tolerance $\delta = 0.01h$ on the condition $\tilde{\alpha}_I(\hat{T}) = q_{max}$, the first step in Fig. 1 converges to $\hat{T} = T^* = 12.8h$ in 10 iterations. The resulting values of $\bar{F}^*$ are represented by the green dashed line in Fig. 2 while the demand profiles and the proportional constraint $\bar{\alpha}^*$ as functions of the measure q are shown in Fig. 3. Note that $\bar{F}^*$ replaces Λ_f as power density of task durations when the reshaped flexible demand $\bar{D}_f = \bar{D}_r$ is considered. Since $\bar{F}^*$ is never greater than Λ_{D_i}, we can conclude that the original equilibrium condition (8) is satisfied when the constraint $\bar{\alpha}$ is enforced. Alternatively consider that, by monotonicity of Q_{D_i}, the inflexible demand function $\bar{D}_i(q) = Q_{D_i}^{-1}(q)$ is always monotonic increasing. The opposite holds for $\bar{D}_f(q)$ which is equal to $\bar{D}_r(q)$ in (27), i.e. the integral of the positive quantity $\bar{F}$ over the interval $[q, T]$. If the sum of the two profiles, equal to the aggregate demand in the variable q, is nondecreasing (like in the present case) an equilibrium is achieved from Proposition 2. Three different intervals, for decreasing values of q, can be considered in Fig. 2 and 3. In particular, for $q > T^*$, it can be seen that $\bar{F}^*$ and the flexible demand $\bar{D}_f$ are equal to zero since all the appliances have already completed their tasks. In the interval which goes from approximatively $6h$ to T^*, the input $\bar{F}^*$ corresponds to the function $\Lambda_f(q - \Delta)$ with Δ equal to about $2h$. In this case the constraint $\bar{\alpha}^*$ is equal to 1 and the resulting demand profile corresponds to the one obtained if all task times were increased by Δ. At about $q = 6h$ the function $\bar{F}^*$ intersects Λ_{D_i}, implying that a constraint $\bar{\alpha}^*(q) < 1$ must be enforced on the power rate of the appliances. This is done by setting $\bar{F}^*(q) = \Lambda_{D_i}(q) = \bar{D}_i'(q)$ which corresponds to a flat profile of aggregate demand in the q variable. The demand profiles and the constraint α^* across time are shown in Fig. 4. It can be seen that a proportional limitation is imposed during the first hours of the day, when inflexible demand (and expected energy prices) are particularly low. Note that the proposed control scheme returns a constraint $\alpha(t)$ which achieves a consistent valley filling, with a flat profile of aggregate demand. To verify that a Nash equilibrium has been indeed achieved, we show in Fig. 5 the scheduled interval of power consumption for devices with t_{min} equal to $2h$, $4h$ and $8h$. These are represented by the blue, green and magenta areas, respectively. Note that each group of devices operates at maximum power consumption at the lowest values of broadcast demand (the inflexible profile D_i), which also correspond to the lowest values of aggregate demand D_a. This implies that no individual agent has interest in changing its scheduled power profile. We also point out that, given the introduction of the proportional constraint α, the intervals of scheduled power consumption are slightly longer that the minimum achievable value t_{min}, as devices cannot always operate at full power rate.

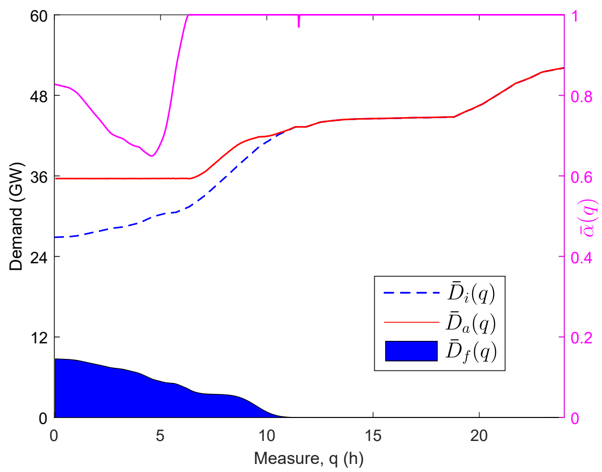

Fig. 3. Profiles of inflexible (blue dashed trace), flexible (blue area) and aggregate demand (red trace) as functions of the measure q when the proportional constraint $\bar{\alpha}^*(q)$ (magenta trace) is imposed.

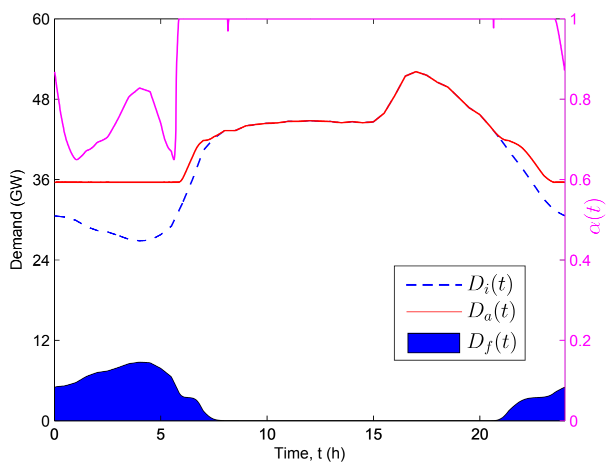

Fig. 4. Profiles of inflexible (blue dashed trace), flexible (blue area) and aggregate demand (red trace) as functions of time when the proportional constraint $\alpha^*(t)$ (magenta trace) is imposed.

Finally, we compare the proposed distributed control strategy with other techniques that have a higher level of centralization. One possibility is to perform a Semi-Decentralized scheduling (**SD**): a Nash equilibrium can still be obtained by an hybrid scheme where all devices independently schedule their power consumption on the basis of broadcast demand/price (with no proportional constraints), introducing central coordination on time intervals where such signal is flat. We also consider the Social Optimum case (**SO**), where the total power required by the appliances is centrally determined and allocated across time in order to minimize generation

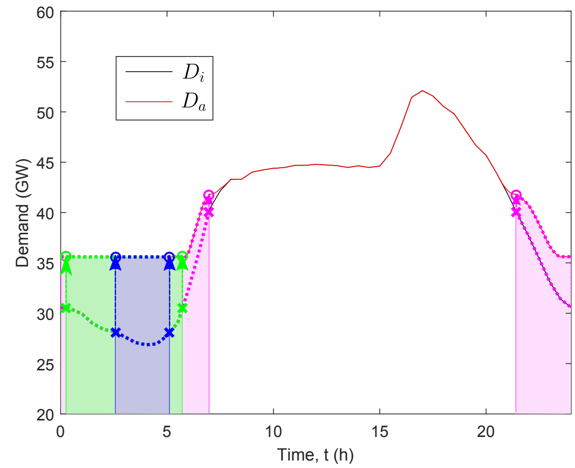

Fig. 5. Intervals of scheduled power consumption for devices with t_{min} equal to $2h$ (blue area), $4h$ (green area) and $8h$ (magenta area).

costs. The aggregate demand profiles obtained in these three cases are shown, for the time interval with lowest values of inflexible demand D_i, in Fig. 6. A clear trade-off can be noticed: the higher is the coordination between appliances, the more significant the induced valley-filling will be. Notice for example that, if a fully centralized algorithm is applied (**SO** case), the aggregate demand profile is completely flat. Similar considerations can be made for total generation costs: it is to be expected that, if all appliances are centrally managed, the total cost of producing energy for the system can be reduced. On the other hand, one must take into account that centralized techniques are unfair towards individual agents, whose utility may be "sacrificed" for the optimality of some global index. Moreover, their application could require additional communication infrastructure and technical solutions that may be more expensive and difficult to implement.

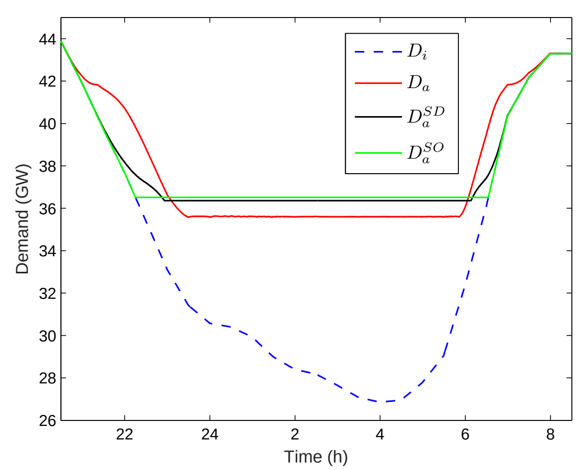

Fig. 6. Comparison of aggregate demand profiles and valley-filling achieved with the proposed power saturation strategy (D_a), semi-decentralized scheduling (D_a^{SD}) and central maximization of social optimum (D_a^{SO}).

VI. Minimization of Individual Task Time

The choice to design the constraint α which induces a Nash equilibrium by minimizing the total operation time of the population can be further justified by showing that the task time of the individual device is also minimized. Recalling the definition of task time $\Gamma(t_{min})$ provided in (18) and assuming that the decentralized control is implemented by designing the function $\bar{F}$ as discussed in previous sections, the problem of minimizing the time required by the single appliance (with parameter $t_{min} = s$) to complete its task can be written as:

$$\begin{array}{rll} \displaystyle\min_{\bar{F}(\cdot),F_{tot},T_{END},T_s} & T_s & \\ \text{s. t.} & \bar{\alpha}_I(0) = 0 & \bar{F}_I(0) = 0 \\ & \bar{\alpha}_I(T_s) = s & T_{END} \leq T \\ & \bar{\alpha}_I(T_{END}^-) = q_{max} & \bar{F}_I(T_{END}^-) = F_{tot} \\ & \dot{\bar{\alpha}}_I(q) = \dfrac{F_{tot} - \bar{F}_I(q)}{\int_{\bar{\alpha}_I(q)}^{T} \frac{f'(\tau)}{\tau}\, d\tau} & \dot{\bar{F}}_I(q) = F(q) \\ & 0 \leq \dot{\bar{\alpha}}_I(q) \leq 1 & \bar{F}(q) \leq \bar{D}_i'(q) \\ & \forall q \in [0, T_{END}) & \end{array} \tag{45}$$

Similarly to (31), the considered constraints guarantee that an equilibrium is achieved for the resulting aggregate demand. Furthermore, condition $\bar{\alpha}_I(T_{END}^-) = q_{max}$ with $T_{END} \leq T$ is introduced to impose that the tasks of the whole population are completed within the considered time interval $[0,T]$. The power absorption of the single device with $t_{min} = s$, in the q variable, is performed in the interval characterized by $\bar{\alpha}_I(q) < s$. Considering the monotonicity of $\bar{\alpha}_I(q)$, it is sufficient to minimize T_s such that $\bar{\alpha}_I(T_s) = s$. It is now possible to verify that the solutions of the task time minimization for the single device and for the whole population (problem (45) and (31), respectively) coincide:

Theorem 2: The control $\bar{F}^*$ defined in (43) and optimal for the problem (31) of global task time minimization, is the solution of (45) for all $s \in \mathscr{T} = [q_{min}, q_{max}]$.

Proof: If $\bar{F}^*$ is not optimal for (45), there exists another control $\bar{F}^\diamond$ which is feasible and such that, for the corresponding solution $(\bar{\varphi}_\alpha^\diamond, \bar{\varphi}_F^\diamond)$ of (28), it holds:

$$\bar{\varphi}_\alpha^\diamond(T_s) = s > \bar{\varphi}_\alpha^*(T_s) \tag{46}$$

where $\bar{\varphi}^*$ denotes the solution of (28) when $\bar{F}^*$ is applied. Since $\bar{\varphi}_\alpha^\diamond(T_{END}^-) = q_{max}$ it is possible to apply Proposition 3 and calculate the equivalent control $\tilde{F}^\diamond$ for (32) to which corresponds the limiting solution $(\tilde{\varphi}_\alpha^\diamond, \tilde{\varphi}_F^\diamond)$ when ε tends to zero. The same operations are performed for the control $\bar{F}^*$ with final time T^*, obtaining the corresponding $\tilde{F}^*$ and the solution $(\tilde{\varphi}_\alpha^*, \tilde{\varphi}_F^*)$ for system (32). Considering the relationship (34) between trajectories in the two systems and the inequality in (46) yields:

$$\tilde{\varphi}_\alpha^\diamond(T_{END} - T_s) < \tilde{\varphi}_\alpha^*(T^* - T_s) \quad \tilde{\varphi}_\alpha^\diamond(T_{END}) = \tilde{\varphi}_\alpha^*(T^*) = q_{max} \tag{47}$$

with $T^* \leq T_{END}$ given the optimality of $\bar{F}^*$ for (31). Furthermore, $\tilde{\varphi}_I(q)$ defined in (54) can be considered as a strictly monotonic increasing function of $\tilde{\varphi}_\alpha(q)$. Denoting the value of $\tilde{\varphi}_I$ when $\tilde{F}^\diamond$ and $\tilde{F}^*$ are considered with the same superscript, it holds:

$$\begin{gathered} \tilde{\varphi}_I^\diamond(T_{END} - T_s) < \tilde{\varphi}_I^*(T^* - T_s) \\ \tilde{\varphi}_I^\diamond(T_{END}) = \tilde{\varphi}_I^*(T^*) = \int_0^{q_{max}} f'(\tau)\, d\tau. \end{gathered} \tag{48}$$

It follows from $T^* \leq T_{END}$ and $\tilde{\varphi}_\alpha^*(0) = 0$ that $\tilde{\varphi}_\alpha^\diamond(T_{END} - T^*) \geq \tilde{\varphi}_\alpha^*(0)$. Given the inequality in (47) and the continuity of the solutions of system (32) there must exist $y \in [T_s, T^*]$ such that:

$$\tilde{\varphi}_\alpha^\diamond(T_{END} - y) = \tilde{\varphi}_\alpha^*(T^* - y) \quad \dot{\tilde{\varphi}}_\alpha^\diamond(T_{END} - y) < \dot{\tilde{\varphi}}_\alpha^*(T^* - y) \tag{49}$$

where the second condition in (49) corresponds to $\tilde{\varphi}_F^\diamond(T_{END} - y) < \tilde{\varphi}_F^*(T^* - y)$ from (32). Taking into account the monotonicity of system (32) with $\hat{T} = y$ and the properties of $\tilde{F}^*$, considering as initial states $\tilde{\varphi}^\diamond(T_{END} - y)$ and $\tilde{\varphi}^*(T^* - y)$, it holds:

$$\tilde{\varphi}_F^\diamond(T_{END} - T_s) \leq \tilde{\varphi}_F^*(T^* - T_s). \tag{50}$$

From conditions (48) on the integral $\tilde{\varphi}_I$, there must exist an interval of positive measure $\mathscr{T} \subseteq [0, T_s]$ such that:

$$\tilde{\varphi}_F^\diamond(T_{END} - \tau) > \tilde{\varphi}_F^*(T^* - \tau) \ \forall \tau \in \mathscr{T}. \tag{51}$$

Given the monotonicity of system (32), established in Proposition 5, and the fact that the control $\tilde{F}^*$ always maximizes the state derivatives, we can conclude that this is not possible. As a consequence, there is no $\bar{F}^\diamond$ for which (46) holds and therefore $\bar{F}^*$ is optimal for (45). ■

VII. Conclusions

In this paper a novel decentralized control strategy is presented for the integration of large populations of price-responsive appliances in the energy market. The conditions presented in [6] for existence and uniqueness of a Nash equilibrium are extended to consider a wider range of scenarios. This is achieved by introducing a proportional constraint on the power rate of the devices, limiting the flexible demand at critical time instants. Such constraint is designed in order to minimize the total time required by the appliances population to complete their tasks and, at the same time, induce a Nash equilibrium. Simulation results are provided and it is shown that the proposed control strategy also minimizes the task time of each individual appliance.

Appendix A
Proof of Proposition 3

The equality for $\bar{\varphi}_F$ and $\tilde{\varphi}_F$ is straightforward to verify:

$$\begin{aligned} \bar{\varphi}_F(\hat{T} - q) &= \int_0^{\hat{T}-q} \bar{F}(\tau)\, d\tau = F_{tot} - \int_{\hat{T}-q}^{\hat{T}} \bar{F}(\tau)\, d\tau \\ &= F_{tot} - \int_0^q \bar{F}(T - \bar{\tau})\, d\bar{\tau} = F_{tot} - \int_0^q \tilde{F}(\bar{\tau})\, d\bar{\tau} = F_{tot} - \tilde{\varphi}_F(q). \end{aligned} \tag{52}$$

For the condition on the states $\bar{\varphi}_\alpha$ and $\tilde{\varphi}_\alpha$, we preliminarily calculate $\tilde{\varphi}_I^\varepsilon(q)$, corresponding to the integral over $[0,q]$ of the solution $\tilde{\varphi}_F^\varepsilon$:

$$\begin{aligned}\tilde{\varphi}_I^\varepsilon(q) &= \int_0^q \tilde{\varphi}_F^\varepsilon(\tau)\,d\tau \overset{a}{=} \int_0^q \dot{\tilde{\varphi}}_\alpha^\varepsilon(\tau)h(\tilde{\varphi}_\alpha^\varepsilon(\tau))\,d\tau \\ &\overset{b}{=} [\tilde{\varphi}_\alpha^\varepsilon(\tau)h(\tilde{\varphi}_\alpha^\varepsilon(\tau))]_0^q - \int_0^q \tilde{\varphi}_\alpha^\varepsilon(\tau)h'(\tilde{\varphi}_\alpha^\varepsilon(\tau))\,d\tau \\ &\overset{c}{=} [\tilde{\varphi}_\alpha^\varepsilon(q)h(\tilde{\varphi}_\alpha^\varepsilon(q)) - \varepsilon h(\varepsilon)] - \int_{v(\tilde{\varphi}_\alpha^\varepsilon(q))}^{v(\varepsilon)} v(\bar{\tau})\frac{f'(\bar{\tau})}{\bar{\tau}}\,d\bar{\tau} \\ &\overset{d}{=} [\tilde{\varphi}_\alpha^\varepsilon(q) - q_{max}]\int_{v(\tilde{\varphi}_\alpha^\varepsilon(q))}^{v(\varepsilon)} \frac{f'(\tau)}{\tau}\,d\tau + [\tilde{\varphi}_\alpha^\varepsilon(q) - \varepsilon]\,h(\varepsilon) \\ &+ \int_{v(\tilde{\varphi}_\alpha^\varepsilon(q))}^{v(\varepsilon)} f'(\tau)\,d\tau\end{aligned} \tag{53}$$

where $v(x) = q_{max} - x$. Equality a can be verified by multiplying $\dot{\tilde{\alpha}}_I^\varepsilon$ in (32) by $h(\tilde{\alpha}_I^\varepsilon)$ and evaluating the result for $\tilde{\alpha}_I^\varepsilon = \tilde{\varphi}_\alpha^\varepsilon$ and $\tilde{F}_I^\varepsilon = \tilde{\varphi}_F^\varepsilon$. Equality b follows from the application of the chain rule and c is obtained with the change of variable $\bar{\tau} = q_{max} - \tilde{\varphi}_\alpha^\varepsilon(\tau)$ in the integral. Rearrangement of the resulting terms yields d. When ε tends to zero the corresponding integral $\tilde{\varphi}_I(q)$ has the following expression:

$$\begin{aligned}\tilde{\varphi}_I(q) = \lim_{\varepsilon\to 0}\tilde{\varphi}_I^\varepsilon(q) = [\tilde{\varphi}_\alpha(q) - q_{max}]\int_{q_{max}-\tilde{\varphi}_\alpha(q)}^{q_{max}} \frac{f'(\tau)}{\tau}\,d\tau \\ + \int_{q_{max}-\tilde{\varphi}_\alpha(q)}^{q_{max}} f'(\tau)\,d\tau.\end{aligned} \tag{54}$$

The integral $\tilde{\varphi}_I(q) = \int_{0^+}^q \tilde{\varphi}_F(\tau)\,d\tau$ can alternatively be evaluated as:

$$\begin{aligned}\tilde{\varphi}_I(q) &\overset{a}{=} \int_{0^+}^q F_{tot} - \bar{\varphi}_F(\hat{T}-\tau)\,d\tau \\ &\overset{b}{=} \int_{0^+}^q \left[\dot{\bar{\varphi}}_\alpha(\hat{T}-\tau)\int_{\bar{\varphi}_\alpha(\hat{T}-\tau)}^{q_{max}} \frac{f'(s)}{s}\,ds\right]d\tau \\ &\overset{c}{=} \int_{\hat{T}-q}^{\hat{T}^-}\left[\dot{\bar{\varphi}}_\alpha(\bar{\tau})\int_{\bar{\varphi}_\alpha(\bar{\tau})}^{q_{max}} \frac{f'(s)}{s}\,ds\right]d\bar{\tau}.\end{aligned} \tag{55}$$

Equality a follows from (52) while b can be verified by multiplying $\dot{\bar{\alpha}}_I$ in (28) by the denominator of its right-hand side, evaluating the result for $\bar{\alpha}_I = \bar{\varphi}_\alpha$ and $\bar{F}_I = \bar{\varphi}_F$. Finally, c is the result of the change of variable $\bar{\tau} = \hat{T} - \tau$. With algebraic passages similar to (53), the following expression can be derived:

$$\tilde{\varphi}_I(q) = -\bar{\varphi}_\alpha(\hat{T}-q)\int_{\bar{\varphi}_\alpha(\hat{T}-q)}^{q_{max}} \frac{f'(\tau)}{\tau}\,d\tau + \int_{\bar{\varphi}_\alpha(\hat{T}-q)}^{\bar{\varphi}_\alpha(\hat{T}^-)} f'(\tau)\,d\tau. \tag{56}$$

The proof is concluded by noticing that $\tilde{\varphi}_I(q)$ as defined in (54) is a monotonic increasing function of $\tilde{\varphi}_\alpha(q)$ and therefore, considering that $\bar{\varphi}_\alpha(\hat{T}^-) = q_{max}$, the two expressions (54) and (56) are equal if and only if $\bar{\varphi}_\alpha(\hat{T}-q) = q_{max} - \tilde{\varphi}_\alpha(q)$.

Appendix B
Proof of Proposition 4

For the equality on the states $\bar{\varphi}_F$ and $\tilde{\varphi}_F$ we have:

$$\begin{aligned}\tilde{\varphi}_F(\hat{T}-q) &= \int_0^{\hat{T}-q}\tilde{F}(\tau)\,d\tau = F_{tot} - \int_{\hat{T}-q}^{\hat{T}}\tilde{F}(\tau)\,d\tau \\ &= F_{tot} - \int_0^q \tilde{F}(\hat{T}-\tau)\,d\tau = F_{tot} - \bar{\varphi}_F(q).\end{aligned}$$

To check that also the first equation in (36) holds, the integral $\bar{\varphi}_I(q)$ is evaluated in two different ways:

$$\bar{\varphi}_I(q) = \int_0^q F_{tot} - \bar{\varphi}_F(\tau)\,d\tau \qquad\qquad \bar{\varphi}_I(q) = \int_0^q \tilde{\varphi}_F(\hat{T}-\tau)\,d\tau$$

With algebraic steps similar to the ones used in the previous proof it is possible to show that the two expressions are equal if and only if $\tilde{\varphi}_\alpha(\hat{T}-q) = q_{max} - \bar{\varphi}_\alpha(q)$.

Appendix C
Proof of Proposition 6

For any $T_1, T_2 \in [0,T]$ with $T_1 < T_2$, the maximum $\gamma^\varepsilon(T_2)$ has the following expression:

$$\gamma^\varepsilon(T_2) = \Phi_\alpha^{T_2}([\varepsilon,0],T_2) = \Phi_\alpha^{T_1}\left(\Phi^{T_2}([\varepsilon,0],T_2-T_1),T_1\right). \tag{57}$$

To see this, it is sufficient to consider that the only dependency of $\Phi^{\hat{T}}$ from the parameter $\hat{T}$ is given by the maximum value $\bar{D}_i'(\hat{T}-q)$ imposed for $\tilde{F}^*$ in (39) at each time q. To prove the Lipschitz continuity of γ^ε, the following inequalities are considered for $|\gamma^\varepsilon(T_2) - \gamma^\varepsilon(T_1)|$:

$$\begin{aligned}&\left|\Phi_\alpha^{T_1}\left(\Phi^{T_2}([\varepsilon,0],T_2-T_1),T_1\right) - \Phi_\alpha^{T_1}([\varepsilon,0],T_1)\right| \\ &\overset{a}{\le} \left\|\Phi^{T_1}\left(\Phi^{T_2}([\varepsilon,0],T_2-T_1),T_1\right) - \Phi^{T_1}([\varepsilon,0],T_1)\right\|_1 \\ &\overset{b}{\le} K_1\left\|\Phi^{T_2,\varepsilon}([\varepsilon,0],T_2-T_1) - [\varepsilon,0]\right\|_1 \overset{c}{\le} K_1K_2|T_2-T_1|\end{aligned} \tag{58}$$

where K_1 and K_2 are positive constants. Inequality a in (58) follows from the expression of the 1-norm for multidimensional vectors while b derives from the continuous differentiability of the solutions of (32) with respect to the initial conditions [19]. This can be verified by replacing $\tilde{F}^*$ in the expressions of (32) and noticing that the partial derivatives of the result with respect to each state component exist and are continuous (almost everywhere). The last inequality c in (58) is a result of the boundedness of the state derivatives. For the monotonicity of γ^ε consider that, when $\tilde{F}^*$ is applied, both states of (32) are nondecreasing in time. Therefore, following Proposition 5, for any $T_1, T_2 \in [0,T]$ with $T_2 > T_1$ we have:

$$\Phi_\alpha^{T_1}\left(\Phi^{T_2}([\varepsilon,0],T_2-T_1),T_1\right) \ge \Phi_\alpha^{T_1}([\varepsilon,0],T_1) \tag{59}$$

where the two sides of the inequality denote respectively $\gamma^\varepsilon(T_2)$ and $\gamma^\varepsilon(T_1)$ as defined in (57) and (41).

Appendix D
Proof of Theorem 1

The existence of T^* is initially shown. In this respect, consider an arbitrary feasible control $\bar{F}$ for (31) such that, for the corresponding state trajectory of system (28), it holds $\bar{\varphi}_\alpha(T_{END}^-) = q_{max}$ at some $T_{END} \in [0,T]$. Applying the results of Proposition 3 for $\hat{T} = T_{END}$, it is possible to define $\tilde{F}$ such that for the resulting limiting solution of (32), considering (34) at $q = \hat{T} = T_{END}$, it holds $\tilde{\varphi}_\alpha(T_{END}) = q_{max}$. Given the optimality of $\tilde{F}^*$ for the value function γ^ε defined in (40), we can conclude that $\gamma(T_{END}) \ge q_{max}$. It follows from the continuity and monotonicity of γ presented in Proposition 5 that there exists T^* defined as in the theorem statement. We show now that $\bar{F}^*$ in (43) is feasible for (31), reminding that $\bar{\varphi}^*$ denotes the solution of (28) when $\bar{F}^*$ is applied and $\tilde{\varphi}^*$ is the limiting solution of system (32) when $\tilde{F} = \tilde{F}^*$ and ε tends to zero. The final state condition $\bar{\alpha}_I(T_{END}^-) = q_{max}$ in (31) becomes $\bar{\varphi}_\alpha^*(T^{*-}) = q_{max}$ and is satisfied from Proposition 4 with $\hat{T} = T^*$ and $q = \hat{T}^-$ since $\gamma(T^*) = \tilde{\varphi}_\alpha^*(T^*) = q_{max}$. For the inequalities on the state derivatives, considering expressions (39) and (42), we have:

$$\tilde{\psi}^*(q) \leq \bar{D}_i'(T^* - q) \qquad \frac{\tilde{\varphi}_F^*(q)}{h(\tilde{\varphi}_\alpha^*(q))} \leq 1 \qquad \forall q \in (0, T^*].$$

To verify the second inequality, given the initial conditions of the system, it is sufficient to notice that $\tilde{\psi}^*(q) = \dot{\tilde{\varphi}}_F^*(q) \leq h'(\tilde{\varphi}_\alpha^*(q))$ when $\tilde{\varphi}_F^*(q) = h(\tilde{\varphi}_\alpha^*(q))$. Using Proposition 4 on the solution $\bar{\varphi}^*$ of system (28) when $\bar{F}^*$ is applied, the following holds at any $q \in [0, T^*)$:

$$\begin{aligned} \bar{F}^*(q) = \tilde{\psi}^*(T^* - q) \leq \bar{D}_i'(q) \\ 0 \leq \dot{\bar{\varphi}}_\alpha^*(q) = \frac{F_{tot} - \bar{\varphi}_F^*(q)}{\int_{\bar{\varphi}_\alpha^*(q)}^{q_{max}} \frac{f'(\tau)}{\tau}\, d\tau} = \frac{\tilde{\varphi}_F^*(T^* - q)}{h(\tilde{\varphi}_\alpha^*(T^* - q))} \leq 1. \end{aligned} \tag{60}$$

Finally, to show the optimality of $\bar{F}^*$, we assume that there exists a control $\bar{F}^\diamond$ which is feasible for (31) and such that, for the corresponding state trajectory $\bar{\varphi}$, it holds $\bar{\varphi}_\alpha(T^{\diamond -}) = q_{max}$ with $T^\diamond < T^*$. If this were the case, it would be possible to define the corresponding control $\tilde{F}^\diamond$ for system (32) using the results of Proposition 3. For the same reasons detailed above for the final instant T_{END}, it would hold $\gamma(T^\diamond) \geq q_{max}$ with $T^\diamond < T^*$ which contradicts the definition of T^*.

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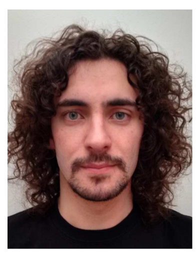

Antonio De Paola (M15) received the M.Sc. degree in control engineering from the University of Rome Tor Vergata, Rome, Italy, in 2011, and the Ph.D. degree in electrical and electronic engineering from Imperial College London, London, U.K., in 2015. He is currently a Research Associate with Imperial College London. His current research interests include optimal control, game theory, and their application to the smart grid.

David Angeli (F15) received the Laurea and Ph.D. degrees in computer science engineering from the University of Florence, Florence, Italy, in 1996 and 2000, respectively. He was an Assistant Professor with the Department of Systems and Computer Science, University of Florence, where he has been an Associate Professor since 2005. He is currently a Reader in Stability of Nonlinear Systems with Imperial College London, London, U.K. His current research interests include stability and dynamics of nonlinear systems and networks, control of constrained systems, and biomolecular dynamics and control solutions for the smart grid. Dr. Angeli served as an Associate Editor of the IEEE Transactions on Automatic Control and Automatica.

Goran Strbac (M95) received the M.Sc. and Ph.D. degrees in electrical energy systems from the University of Belgrade, Belgrade, Serbia, in 1989 and 1994, respectively. He is currently a Professor of Electrical Energy Systems with Imperial College London, London, U.K. His current research interests include modeling and optimization of electricity system operation and investment, economics, and pricing, including the integration of new forms of generation and demand-side technologies.